

# CLOSED FORM SOLUTION FOR MINIMUM NORM MODEL-VALIDATING UNCERTAINTY

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## Abstract

A methodology in which structured uncertainty models are directly constructed from measurement data for use in robust control design of multivariable systems is proposed. The formulation allows a general linear fractional transformation uncertainty structure connections with respect to a given nominal model. Existence conditions are given, and under mild assumptions, a closed-form expression for the smallest norm structured uncertainty that validates the model is given. The uncertainty bound computation is simple and is formulated for both open and closed loop systems.

## 1. Introduction

Currently available system identification (ID) techniques are well-developed under ideal conditions of finite dimensionality and linear, time-invariant (LTI) system as given in references such as [1, 2]. “Robust” control under this ideal case amounts to a classical sensitivity reduction (or disturbance rejection) problem due to unknown but bounded disturbance. Possible exceptions for introducing model uncertainty for LTI system is during model reduction for lower order controller design (for example [3, 4, 5]), or when the quality and quantity of measurement data available are in question.

For many situations in engineering a finite dimensional LTI system is only an approximation of a true plant. There is no need to justify the frequent occurrence of the above discrepancies which are possibly due to an unknown combination of physical causes. Often, the need for robust control arises due to a suspected “corruption” of available measurement data by the secondary effects of nonlinearities and/or time variations for approximately LTI systems. Although strictly speaking, robust control theory for nonlinear, time-varying systems should be applied under the above circumstances, it is currently limited. However, it has been shown that LTI-based robustness theory can handle a class of slowly time varying and nonlinear uncertainties or effects via conic sector theory [6, 7, 8]. The hope is that a small set of LTI plants will be sufficient to encompass these secondary effects.

This paper is an attempt to methodically construct realistic uncertainty models directly from input output measurements, to make a body of multivariable robust control theory work in real applications. It extends the open loop robust ID result in [9] to a general closed loop setting. In section 2, we briefly describe a the form of the structured uncertainties typically assumed in robust control analysis and design and in this paper. The main result is outlined

in section 3, where the input and output residuals about a nominal model is related to the uncertainties allowed about the nominal plant in a generic closed loop system ID setting. Model validation and minimum norm model validation problems are defined. Conditions for the existence of a model validating solution are given. A closed form expression for the minimum norm uncertainty among all model-validating structured uncertainty is also given. As a special case, robust identification in the open loop case is shown in Section 4. Section 5 gives concluding remarks.

## 2. Uncertainty Structure

Let the overall structured uncertainty be defined by the block diagonal matrices

$$\Delta = \text{diag}(\Delta_1, \dots, \Delta_\tau); \quad \Delta_j \in C^{m_j \times n_j} \quad (1)$$

and the set of all block diagonal and stable, rational transfer function matrices be given by

$$\mathcal{D} = \{\Delta(\cdot) \in RH_\infty : \Delta_j(s_o) \in C^{m_j \times n_j}, \forall s_o \in \bar{C}_+\} \quad (2)$$

where  $\tau$  and  $\bar{C}_+$  denote the number of uncertainty blocks and the closed right-half plane, respectively [11]. We consider the class of problems where the uncertainty connections to the nominal and the plant inputs and outputs are given. The relationship between the plant input and output can be written as

$$y = F_u(P, \Delta)u + \tilde{n} \quad (3)$$

where  $\tilde{n}$  denotes a noise vector whose spectrum is assumed known and the upper linear fractional transformation (LFT) is defined by

$$F_u(P, \Delta) = P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12} \quad (4)$$

and  $P$  denotes the augmented plant. Notice that in conventional system ID, only  $P_{22}$  is of interest and  $\Delta = 0$ . The important point is that if  $P_{22}$ , i.e., the nominal plant is known, the rest of the augmented plant can be constructed from a priori assumptions on the uncertainty connections based on engineering judgement of the physical system. The structured uncertainty freedom allows the envelopment of the scatter or variation in the input output data by a bound on  $\Delta$  about the nominal. In the next section, uncertainty models are used to bound residuals that remain after a nominal model fit.

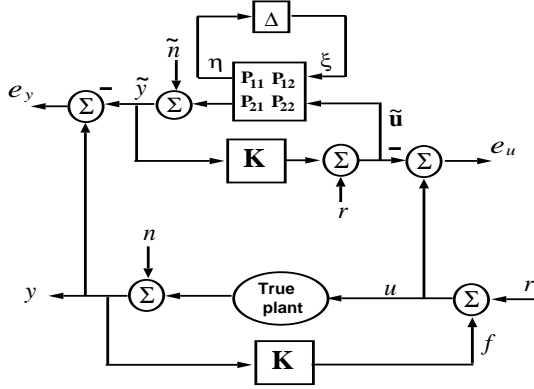
### 3. Closed Loop Robust ID

### 3.1. Residual Errors

Given a  $P - \Delta$  structure and estimated output noise,  $\tilde{n}$ , the errors

$$e := \begin{Bmatrix} e_y \\ e_u \end{Bmatrix} = \begin{Bmatrix} y \\ u \end{Bmatrix} - \begin{Bmatrix} \tilde{y} \\ \tilde{u} \end{Bmatrix} \quad (5)$$

between the predicted (denoted by  $\hat{\cdot}$ ) and measured values of the plant outputs and inputs can be related directly to the structured uncertainty. Figure 1 show these errors for



**Figure 1:** General block diagram for robust ID

a general closed loop system ID experiment. It is assumed that the controller,  $K$ , and external command,  $r \in R^{n_r}$ , are known and the plant inputs,  $u \in R^{n_u}$ , and outputs,  $y \in R^{n_y}$ , are measured. The fictitious signals have dimensions  $\eta \in C^{n_\eta}$  and  $\xi \in C^{n_\xi}$  where

$$n_\eta = \sum_{j=1}^{\tau} n_j; \quad n_\xi = \sum_{j=1}^{\tau} m_j \quad (6)$$

The symbols,  $R^{n_u}$  and  $C^{n_\eta}$  denote real and complex vector spaces of dimension  $n_u$ , and  $n_\eta$  respectively. For the case when external disturbance at the plant input is present, it can be modeled approximately by adding a filtered disturbance at the output.

The predicted outputs and inputs are given by

$$\begin{Bmatrix} \tilde{y} \\ \tilde{u} \end{Bmatrix} = T(\Delta) \begin{Bmatrix} r \\ \tilde{n} \end{Bmatrix} \quad (7)$$

where

$$\mathbf{T}(\Delta) = \begin{bmatrix} (I - F_u(P, \Delta)K)^{-1}F_u(P, \Delta) & (I - F_u(P, \Delta)K)^{-1} \\ (I - KF_u(P, \Delta))^{-1} & (I - KF_u(P, \Delta))^{-1}K \end{bmatrix} \quad (8)$$

Note that  $T$  represents the four components of transfer matrices that define internal stability of a general two block feedback system consisting of  $F_u(P, \Delta)$  and  $K$ . Since the closed loop ID experiment is internally stable, the system consisting of  $F_u(P, \Delta)$  and  $K$  is assumed to be stable also. This translates to robust stability of  $(P, K)$  with respect to  $\Delta$ .

In order to solve for the uncertainty, it will be more convenient to rewrite matrix  $T(\Delta)$  in equation 8 so that

$\Delta$  appears as an argument in an LFT. This useful form can be summarized as follows:

*Lemma 1:*

$$\left\{ \begin{array}{c} \tilde{y} \\ \tilde{u} \end{array} \right\} = F_u(R, \Delta) \left\{ \begin{array}{c} r \\ \tilde{n} \end{array} \right\} \quad (9)$$

where

$$R = \left[ \begin{array}{cc} F_L(P, K) & P_{12}(I - K P_{22})^{-1} [I \ K] \\ \left[ \begin{array}{c} I \\ K \end{array} \right] (I - P_{22} K)^{-1} P_{21} & T(0) \end{array} \right] \square \quad (10)$$

*Proof of Lemma 1:* Consider the basic relations:

$$\eta = P_{11}\xi + P_{12}\tilde{u} \quad (11)$$

$$\tilde{y} = P_{21}\xi + P_{22}\tilde{u} + \tilde{n} \quad (12)$$

$$\tilde{u} = K\tilde{y} + r \quad (13)$$

$$\xi = \Delta\eta \quad (14)$$

Equations 11 to 13 can be rearranged to

$$\begin{aligned} & \begin{bmatrix} I & 0 & -P_{12} \\ 0 & (I - P_{22}K) & 0 \\ 0 & 0 & (I - KP_{22}) \end{bmatrix} \begin{Bmatrix} \eta \\ \tilde{y} \\ \tilde{u} \end{Bmatrix} \\ &= \begin{bmatrix} P_{11} & 0 & 0 \\ P_{21} & P_{22} & I \\ KP_{21} & I & K \end{bmatrix} \begin{Bmatrix} \xi \\ r \\ \tilde{n} \end{Bmatrix} \quad (15) \end{aligned}$$

Using the partitioned matrix inverse identity the coefficient matrix in the left hand side of equation 15 can be inverted so that

$$\begin{Bmatrix} \eta \\ \tilde{y} \\ \tilde{u} \end{Bmatrix} = R \begin{Bmatrix} \xi \\ r \\ \tilde{n} \end{Bmatrix} \quad (16)$$

Equations 14 and 16 gives 9.  $\square$

$T(0)$  denotes the nominal value of  $T(\Delta)$  in equation 8

$$T(0) = \begin{bmatrix} (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \\ (I - KP_{22})^{-1} & (I - KP_{22})^{-1}K \end{bmatrix} \quad (17)$$

Note that  $T(0) = F_u(R, 0)$  corresponds to the four component transfer function matrices of the two block nominal feedback system. Hence, internal stability of the nominal closed loop system is equivalent to stability of  $T(0)$ .

Using Lemma 1, the error in equation 5 is written as

$$e = e_o - R_{21}\Delta(I - R_{11}\Delta)^{-1}M_{12} \quad (18)$$

where

$$e_o = \begin{Bmatrix} y \\ u \end{Bmatrix} - T(0) \begin{Bmatrix} r \\ \tilde{n} \end{Bmatrix} \quad (19)$$

$$M_{12} = R_{12} \left\{ \begin{array}{c} r \\ \tilde{n} \end{array} \right\} \quad (20)$$

Note that  $e_o$  is the residual from nominal fit, i.e., when  $\Delta = 0$ .

### 3.2. Model Validation

Equation 18 gives the residual error that remains after a nominal model fit of the available measurement data. In applications, measurement data are usually discrete and finite length in time which will lead to a spectrum (from spectral sampling theory) at evenly spaced discrete frequencies. The discrete frequencies will be denoted by the set,  $\Omega$ , where

$$\Omega = (z_1, \dots, z_{n_\omega}); \quad z_i = e^{j\omega_i h} \quad (21)$$

where  $h$  denotes the sampling time. The errors defined in equation 18 are computed at the above frequencies by taking the discrete Fourier transform of both discrete time signals and systems.

#### Model Validation:

Given the time histories,  $u$ ,  $y$ ,  $r$ , controller,  $K$ , and an augmented plant,  $P$ , the model is said to be validated if there exists  $\Delta \in \mathcal{D}$  such that  $e = 0$  for all frequencies  $\Omega$ , i.e.,

$$R_{21}\Delta(I - R_{11}\Delta)^{-1}M_{12} = e_o; \quad \forall z_i \in \Omega \quad \square \quad (22)$$

The above definition for model validation requires the existence of the uncertainty matrix,  $\Delta$ , such that the predicted values match the measurement data. The use of the uncertainty matrix extends the more traditional viewpoint of model validation where the question involves the matching of a single plant model with measurement data. If a model validating  $\Delta$  exists at a frequency, its magnitude is a reflection of the fit error of the nominal model from the given measurement data at that frequency.

A necessary condition for a solution to the model validation problem is if there exists,  $\psi \in C^{n_y}$  such that

$$\begin{bmatrix} I \\ K \end{bmatrix} \psi = e_o \quad (23)$$

Of course it is also sufficient if  $\psi$  and  $\Delta$  satisfies

$$\psi = (I - P_{22}K)^{-1}P_{21}\Delta(I - R_{11}\Delta)^{-1}M_{12} \quad (24)$$

Interestingly, the structure of the problem leads to the fact that a solution  $\psi$  always exists that satisfies equation 23. This fact can be proven as follows. First, equation 23 is written as

$$\psi = e_o^y \quad (25)$$

$$K\psi = e_o^u \quad (26)$$

where

$$e_o := \begin{Bmatrix} e_o^y \\ e_o^u \end{Bmatrix} \quad (27)$$

Therefore, the existence of a solution,  $\psi$ , to equation 23 reduces to satisfaction of the condition

$$Ke_o^y = e_o^u \quad (28)$$

The above condition is always satisfied since

$$e_u := u - \tilde{u} = Ke_y \quad (29)$$

for any  $\Delta$ . Observe that when the output residual is zero, the input residual also becomes zero. Indeed it is then not

surprising to find in the sequel that the model validation conditions for a closed loop system is similar to an open loop system.

The closed loop model validation problem reduces to satisfying the following condition from equations 24 and 25:

$$e_o^y = (I - P_{22}K)^{-1}P_{21}\Delta(I - R_{11}\Delta)^{-1}M_{12}, \quad \forall z_i \in \Omega \quad (30)$$

By assuming that the output sensitivity of the nominal closed loop transfer function matrix do not have transmission zeros on the unit circle, the above equation can be simplified to the form

$$P_{21}\Delta(I - R_{11}\Delta)^{-1}M_{12} = (I - P_{22}K)e_o^y, \quad \forall z_i \in \Omega \quad (31)$$

Notice that for closed loop robust ID, the nominal model,  $P_{22}$ , influence both the nominal residual error and the uncertainty effect on model validation on the left hand side of equation 31.

We formally define the problem of finding minimum norm uncertainty that satisfies model validation.

#### Minimum Norm Model Validation (MNMV):

Given residual errors after the best nominal model fit,  $e_o^y$ , for  $\forall z_i \in \Omega$ , find the smallest norm uncertainty  $\bar{\sigma}(\Delta)|\Delta \in \mathcal{D}$  among the model validating uncertainties that satisfy Eq.(31).  $\square$

### 3.3. Solution of MNMV Problem

The algorithm presented here finds a smallest possible bound  $\delta$  at each frequency  $z_i \in \Omega$  such that a  $\Delta \in \mathcal{D}$  exists and satisfies equation 31. The optimal value of  $\delta$  would come from finding a minimum norm solution to equation 31. In this section, a technique is given which, under some reasonably mild assumptions, gives the minimal norm solution to Eq.31 without resorting to optimization via non-linear programming.

Since the determination of minimum norm bound is independent at each frequency, the underlying frequency will be fixed for the remainder of this section. Also in this section, the following assumptions are made:

*Assumption 1:*  $n_\xi \geq n_y$

*Assumption 2:*  $P_{21}$  is full rank.

*Assumption 3:*  $\Delta$  satisfies equation 1.

*Assumption 4:*  $I - R_{11}\Delta$  is invertible.

Assumption 1 states that the number of uncertainty freedoms is at least as large as the number of output channels. This along with the rank condition in assumption 2 ensures that a model validating uncertainty exists. Assumptions 1 and 2 are made for convenience since we only need to satisfy

$$(I - P_{22}K)e_o^y \in \text{Range}(P_{21}) \quad (32)$$

Physically, this is a requirement that the weighted output residual from nominal fit must be within the domain of influence of the uncertainty freedom as given by the range space of  $P_{21}$ .

Assumption 3 simply is part of the definition of structured uncertainty. The invertibility condition in assumption 4 is equivalent to the well-posedness condition of the two block loop involving  $R_{11}$  and  $\Delta$ . Since  $R_{11} = F_l(P, K)$  and  $\Delta$  is the structured uncertainty, this assumption is a

necessary condition for robust stability [11] with respect to structured uncertainty, which is also equivalent to internal stability of  $T(\Delta)$  in equation 8. Since closed loop stability is necessary for a succesful system ID experiment and hence robust stability, assumption 4 will likely be satisfied.

Define the singular value decomposition (SVD)

$$P_{21} = USV^* \quad (33)$$

where  $U \in C^{n_y \times n_y}$  and  $V \in C^{n_\xi \times n_\xi}$  are Hermitian matrices and  $S \in R^{n_y \times n_\xi}$  is a full rank diagonal matrix due to assumption 2. Then the Moore-Penrose pseudo-inverse is

$$P_{21}^+ = VS^+U^* \quad (34)$$

where  $S^+$  is a diagonal matrix the same size as  $S^T$ , having on its diagonal the reciprocals of the diagonal elements of  $S$ . Denote by  $\mathcal{N}$  the null space of  $P_{21}$ . It has dimension  $n_\xi - n_y$  and is spanned by the last  $n_\xi - n_y$  columns of  $V$ . Let

$$w = P_{21}^+(I - P_{22}K)e_o^y \quad (35)$$

Then a matrix  $\Delta$  solves Eq.(31) if and only if

$$\Delta(I - R_{11}\Delta)^{-1}M_{12} = w + \phi \quad (36)$$

for some  $\phi \in \mathcal{N}$ . The pseudo-inverse solution (when  $\phi = 0$ ) corresponds to the condition where the Euclidian norm of the fictitious uncertainty signal,  $\xi$ , is minimal. This is clear from equations 14 and 16 where

$$\xi = \Delta(I - R_{11}\Delta)^{-1}M_{12} \quad (37)$$

The minimum norm signal is over all uncertainty signals that validates the given data about the nominal. This signal should be distinguished from the minimum norm uncertainty,  $\Delta$ .

To find  $\Delta \in \mathcal{D}$  which satisfy (36), the identity

$$\Delta(I - R_{11}\Delta)^{-1} = (I - \Delta R_{11})^{-1}\Delta \quad (38)$$

is used to rearrange the LFT in Eq.(36) to the form

$$\Delta(M_{12} + R_{11}(w + \phi)) = w + \phi \quad (39)$$

For simplicity, let

$$x := M_{12} + R_{11}(w + \phi); \quad y := w + \phi \quad (40)$$

Note that  $x$  and  $y$  are functions of  $\phi$ , an arbitrary null vector of  $P_{21}$ , but are otherwise completely determined at the fixed underlying frequency.

Partition

$$x = \text{col}(x_1, \dots, x_\tau); \quad y = \text{col}(y_1, \dots, y_\tau) \quad (41)$$

in a conformal manner with respect to the uncertainty blocks,  $\Delta_1, \dots, \Delta_\tau$  as defined in equation 2. First observe that an SVD for  $\Delta$  can be built by placing corresponding matrices from SVD's for each  $\Delta_i$  along the diagonals of block diagonal matrices. Thus,  $\bar{\sigma}(\Delta) = \max_i \bar{\sigma}(\Delta_i)$ . Next, observe that  $\Delta x = y$  if and only if  $\Delta_i x_i = y_i$  for all  $i = 1, \dots, \tau$ . Then note that if  $\Delta_i x_i = y_i$  with  $x_i \neq 0$ , then

$$\bar{\sigma}(\Delta_i) = \|\Delta_i\|_2 \geq \frac{\|y_i\|_2}{\|x_i\|_2} \quad (42)$$

The following lemma shows that if  $x_i \neq 0$ , a  $\Delta_i$  may be chosen to achieve this lower bound.

*Lemma 2:*

If  $u \in C^m$ ,  $v \in C^n$ ,  $v \neq 0$ , then there exists  $A \in C^{m \times n}$  such that  $Av = u$ , and  $\bar{\sigma}(A) = \frac{\|u\|_2}{\|v\|_2}$ .  $\square$

*Proof of Lemma 2:*

The proof is by construction. Choose  $A$  in the SVD form

$$A = [u/\|u\|_2 \quad U^{\perp_u}] \Sigma [v/\|v\|_2 \quad V^{\perp_v}]^* \quad (43)$$

where  $\Sigma = \text{diag}(k_1, \dots, k_s, 0, \dots, 0)$ ,  $k_1 \geq k_j$ ,  $j = 2, \dots, s$ . The matrices  $U^{\perp_u}$  and  $V^{\perp_v}$  denote an orthonormal bases for the orthogonal complement subspaces of  $u \subset C^m$  and  $v \subset C^n$ , respectively. Then the left hand side of the equation is

$$Av = [u/\|u\|_2 \quad U^{\perp_u}] \Sigma \begin{Bmatrix} \|v\|_2 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad (44)$$

$$= uk_1 \frac{\|v\|_2}{\|u\|_2} \quad (45)$$

Therefore if we choose  $k_1 = \frac{\|u\|_2}{\|v\|_2}$  then  $A$  satisfies the equation and the norm  $\bar{\sigma}(A) = \frac{\|u\|_2}{\|v\|_2}$ .  $\square$

The final assumption is stated as

*Assumption 5:* There exists  $\phi \in \mathcal{N}$  so that for each  $i$ , if  $x_i = 0$ , then  $y_i = 0$ .

Note that assumption 5 is only necessary for the singular condition when  $x_i = 0$ . For each such  $\phi$ , Lemma 2 can be applied to each  $x_i$  and  $y_i$  which are nonzero, to find a minimum norm of  $\Delta_i$  for which  $\Delta_i x_i = y_i$ . If  $y_i = 0$  zero then minimal  $\Delta_i$  is zero. This demonstrates the following result:

*Proposition:*

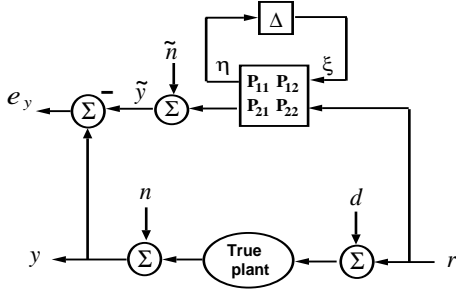
For each  $\phi \in \mathcal{N}$ , there is a  $\Delta \in \mathcal{D}$  with  $\bar{\sigma}(\Delta) \leq \delta_\phi$  which satisfies equation 31 where  $\delta_\phi = \max_i \frac{\|y_i\|_2}{\|x_i\|_2}$ .  $\square$

The simple technique to bound the uncertainty is to calculate  $\delta_\phi$  for any  $\phi \in \mathcal{N}$  for which assumption (5) is satisfied. The obvious first choice to try is  $\phi = 0$ . If  $n_\xi = n_y$ , this is the only  $\phi \in \mathcal{N}$ . The exploitation of the freedom in  $\phi$  is posed as a conventional constrained optimization problem in [9]. This freedom is investigated further in [10].

To summarize, the minimum norm bound for each uncertainty block can be computed from Eq.(42). For robust control design application, the minimum bounds can be overbound by a stable, realizable low-order transfer functions for each uncertainty block such that  $\Delta(z) \in RH_\infty$ . The linear programming approach [12] can be used to obtain a tight overbound with low-order functions.

#### 4. Open Loop Robust ID

The robust ID from open loop experiments can be viewed as a special case of closed loop robust ID. By letting the controller,  $K$  be zero, figure 1 simplifies to figure 2. For



**Figure 2:** Block diagram for open loop robust ID

the open loop case,  $u = \tilde{u} = r$  so that  $e_u = 0$  and the only error to consider is in the predicted outputs. Equations 10 and 16 simplifies to

$$\begin{Bmatrix} \eta \\ \tilde{y} \end{Bmatrix} = R \begin{Bmatrix} \xi \\ r \\ \tilde{n} \end{Bmatrix} \quad (46)$$

where

$$R := \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & [P_{12} & 0] \\ P_{21} & [P_{22} & I] \end{bmatrix} \quad (47)$$

so that the predicted output is

$$\tilde{y} = F_u(R, \Delta) \begin{Bmatrix} r \\ \tilde{n} \end{Bmatrix} \quad (48)$$

The error in the predicted output for open loop is

$$e_y = e_o^y - P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}r \quad (49)$$

where

$$e_o^y = y - P_{22}r - \tilde{n} \quad (50)$$

is the residual after a nominal fit.

The definition of model validation for open loop case is similar to the closed loop case except that  $u = r$ ,  $K = 0$ , and only  $e_y$  is required to be zero, so that the model validation condition reduces to

$$P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}r = e_o^y \quad (51)$$

The condition in equation 51 has a similar form as the closed loop equation 31. However, in the open loop ID case, the nominal residual output error is not premultiplied by the output return difference, and the output noise and nominal model,  $P_{22}$ , only appears in the residuals in the nominal fit in the right hand side of equation 51 (cf 31).

In solving the MNMV problem for the open loop case, the same solution algorithm can be used because equations 31 and 51 have the same form. The only difference is in the physical significance of assumption 4 where it is sufficient that the true plant needs to be stable which is necessary for open loop ID. This means that assumption 4 will always be satisfied in the open loop case.

#### 5. Concluding Remarks

A key assumption is that we can represent the scatter in the input-output measurements by a set of plants defined by a linear fractional transformation. Additionally, an uncertainty structure about the nominal model are judiciously selected by an engineer based on the underlying physics of the problem. The residuals in the predicted inputs and outputs are used to generate bounds of structured uncertainties for each component in the frequency domain. The simplicity and directness of the calculations for the uncertainty bounds is encouraging. Recent applications show reasonable predictions of structured uncertainty bounds for a stable SISO plant [9] in open loop and for an unstable MIMO plant [13] in closed loop.

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